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1978 J. Phys. A: Math. Gen. 11 975

(http://iopscience.iop.org/0305-4470/11/5/027)

J. Phys. A: Math. Gen., Vol. 11, No. 5, 1978. Printed in Great Britain

A real space time-dependent renormalisation group of the one-dimensional Ising model

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Received 12 October 1977, in final form 22 November 1977

Abstract. The critical slowing down, and its dynamic exponent are studied in the one-dimensional Glauber–Ising model. The calculations serve as an example of the application of the ideas of the real space renormalisation group in time-dependent problems.

1. Introduction and discussion

The study of critical phenomena with the aid of the renormalisation group (RG) approach provides a verification of the hypothesis of universality, a way to extract critical exponents and amplitude ratios, as well as a method to calculate thermodynamic and correlation functions. Under the RG transformation (Wilson 1971), the effective Hamiltonian, $\mathcal{H}(l)$, appearing in the partition function, is transformed to $\mathcal{H}(l+1)$ through the change of the space scale, and a rescaling of the appropriate degrees of freedom. In general, it is very difficult to find the transformation R_l, $\mathcal{H}(l+1)=R_l\mathcal{H}(l)$. Thus several methods of approximation were developed. One method is the ϵ expansion (Wilson and Fisher 1972, Wilson and Kogut 1974), in which the short wavelength modes are integrated out in a $d(=4-\epsilon)$-dimensional system. Another method is the real space renormalisation (Niemeijer and van Leeuwen 1976, Kadanoff *et al* 1976, Nelson and Fisher 1975). In the real space renormalisation a new spin variable, μ_i, is defined on a new lattice which has a larger lattice constant. This transformation is of the form

$$\exp\left(-\frac{\mathcal{H}'\{\mu\}}{k_B T}\right)=\sum_{\{\sigma\}}\mathcal{S}\{\mu,\sigma\}\exp\left(-\frac{\mathcal{H}\{\sigma\}}{k_B T}\right), \qquad \sum_{\{\mu\}}\mathcal{S}\{\mu,\sigma\}=1 \tag{1.1}$$

so that the partition function is invariant under the transformation. (k_B is the Boltzmann constant and T is the temperature). If $\mathcal{S}\{\mu,\sigma\}=\Pi_i\,\delta(\mu_i-\sigma_{n_i})$, the transformation is called 'decimation', under which a fraction of the σ are integrated out. Other forms of $\mathcal{S}$ can describe the block transformations in which the μ_i does not coincide with one of the old σ_i, but describes a block of $\{\sigma_i\}$.

The success of the RG approach in the understanding of static critical phenomena motivated the adoption of it in the study of critical dynamics. According to the conventional theory (Van Hove 1954) the lifetime $\tau(k)$ of the k Fourier component of the order parameter is given by

$$\tau(k)=\chi(k)/D$$

where $\chi(k)=\langle\sigma_k\sigma_{-k}\rangle$ is the k-dependent susceptibility of the order parameter, and D is the transport coefficient. D is proportional to k^2 if the order parameter is conserved. This result was generalised by the dynamic scaling hypothesis (Ferrell *et al* 1968, Halperin and Hohenberg 1969) by the definition of the dynamic exponent z such that

$$\tau(k)\sim\xi^z F(k\xi)$$

where ξ is the correlation length and $F(y)$ is some universal function. In the conventional theory the non-conserved order parameter has $z_{\text{conv}}=2-\eta$ where η is the correlation function exponent. The above relation between τ and ξ indicates that under a spatial rescaling the lifetime will also rescale. Thus, the equation of motion will be invariant under the rescaling of the space and the characteristic time. The parameter space includes not only the usual static components but also the bare transport coefficients. The exponent z is determined so that the transport coefficient of the slower mode will not be changed under the RG (Ma 1967a, b).

The first explicit solution of a time-dependent RG was given by Halperin *et al* (1972, 1974). They introduced the time-dependent Ginzburg–Landau model which is the continuum spin version (Myerson 1976) of the Glauber (1963) model for one-spin flip processes combined with the Landau–Ginzburg–Wilson Hamiltonian (Wilson and Kogut 1974). In the continuum spin models, the large k variables represent the rapidly varying modes. Hence the RG transformation can be done by expressing the fast modes in terms of the slow ones. This procedure will renormalise the noise term in the equation of motion, and will change the characteristic cut-off of the problem. Since the coupling between the fast and the slow modes has a fixed point of order ϵ, one can find a systematic way of obtaining the recursion relations as an expansion in powers of ϵ.

Most work on critical dynamics is based on the Ginzburg–Landau formulation and results in expansions around four and six dimensions (Hohenberg and Halperin 1977). Only one attempt has been made to study the dynamics of a model of discrete spins on a lattice. In this work (Ma 1976a, b) a Monte-Carlo method was used for a numerical study of the behaviour of blocks in the two-dimensional Ising model. The results revealed the existence of two time scales. When the block spin flips over, rapid fluctuations appear. The longer time scale describes the collective behaviour of a block. The appearance of the two time scales, and the absence of a formalism cause difficulties in the numerical analysis (Ma 1976a, b), and in the interpretation of the results.

In this article we want to show how the idea of the RG transformation on a lattice can be generalised to time-dependent problems. As an example we choose the one-dimensional Glauber (1963) model which has the advantage of both being simple and of being the only exactly soluble model known. The reader is referred to Nelson and Fisher (1975) for a detailed discussion of the discrete static RG transformation in the one-dimensional Ising model. In § 2 we review the Glauber model and perform the decimation transformation. In § 3 the same model is solved using the block transformation. In the two methods, when the external field H vanishes the exact recursion relations obeyed by the static parameters and the time scale are found. When $H\neq 0$, we have been able to find only the recursion relations valid to linear order in H. The static limit coincides with the known static recursion relations (Nelson and Fisher 1975). However, when $H\neq 0$ one needs to make use of the static RG transformation of quantities calculated when $H=0$. Since it is difficult to identify the fast and the slow modes, we eliminate one of two equivalent modes. This procedure, which is quite obvious in the decimation transformation is generalised in the block transformation. In both transformations it is shown that the RG creates memory effects which are, however, irrelevant in the RG sense.

2. The decimation of the Glauber model

The one-dimensional Glauber (1963) model is a chain of stochastic functions of time $\sigma_i(t)$ which are restricted to the values ± 1 and make transitions randomly between these two values via an interaction with an external heat bath. The equilibrium properties of the system are determined by the Ising Hamiltonian

$$H = -J \sum_i \sigma_i \sigma_{i+1}. \tag{2.1}$$

Glauber assumed that the dynamics of the system is a single spin flip process. Thus, the time derivative of the spin probability functions, $P(\sigma_1 \ldots \sigma_n; t)$, is

$$\frac{\mathrm{d}}{\mathrm{d}t} P(\sigma_1 \ldots \sigma_n; t) = -\sum_j \left(W_j(\sigma_j) P(\sigma_1 \ldots \sigma_n; t) - W_j(-\sigma_j) P(\sigma_1 \ldots -\sigma_j \ldots \sigma_n; t)\right) \tag{2.2}$$

where W_j is the probability of the jth spin to flip in a unit time. Glauber suggested that W_j are of the form:

$$W_j(\sigma_j) = \tfrac{1}{2} a [1 - \tfrac{1}{2}\gamma\sigma_j(\sigma_{j-1} + \sigma_{j+1})] \tag{2.3}$$

where $\gamma = \tanh(2J/k_{\mathrm{B}}T)$ determined by the detailed balance at equilibrium (k_{B} is the Boltzmann constant, T is the temperature, and a which fixes the time scale will be taken to be unity in the following). This choice of W is not unique, but has the advantage of simplicity rather than profound physical background. After few manipulations one can obtain a master equation for the spin average,

$$\begin{aligned} q_i(t) &= \langle \sigma_i(t) \rangle = \sum_{\{\sigma\}} \sigma_i P(\{\sigma\}, t), \\ \mathrm{d}q_i/\mathrm{d}t &= -q_i(t) + \tfrac{1}{2}\gamma(q_{i-1}(t) + q_{i+1}(t)). \end{aligned} \tag{2.4}$$

As this equation stands, it has an ideal form for a real space renormalisation. The $q_{i\pm 1}(t)$ can be solved in terms of q_i and $q_{i\pm 2}$ and so they are eliminated. This procedure is equivalent to a scaling of the space by a factor of two. The reader will notice that in contrast to a more conventional treatment we do not eliminate a rapid mode in favour of a slower mode. Here one of two equivalent modes is eliminated. As a result of the scaling, (2.4) becomes

$$\left(-\mathrm{i}\omega \frac{2}{1-\gamma^2/2} - \frac{\omega^2}{1-\gamma^2/2} + 1\right) q_i(\omega) = \frac{1}{2} \frac{\gamma^2/2}{1-\gamma^2/2} (q_{i-2} + q_{i+2}) \tag{2.5}$$

where

$$q_i(\omega) = \int_{-\infty}^{\infty} q(t) \exp(\mathrm{i}\omega t)\, \mathrm{d}t.$$

In terms of the new parameters,

$$\gamma' = \gamma^2/(2-\gamma^2) \tag{2.6}$$

$$\omega' = \omega(2-\mathrm{i}\omega)/(1-\gamma^2/2) \tag{2.7}$$

equation (2.5) has the same form as before the space scaling was performed. Thus (2.6) and (2.7) are the recursion relations for the interaction and the time scale. Equation (2.6) describes the static RG which is independent of the time renormalisation (as it should be). $\kappa = \exp(2J/k_B T)$, the asymptotic temperature dependence of the correlation length obeys the known exact recursion relation (Nelson and Fisher 1975):

$$\kappa^{-2} = 4\kappa^{-2}/(1+\kappa^{-2})^2,$$

and gives the ferromagnetic fixed point $\xi^{-1} = 0$ at $T = 0$. To extract information on the dynamics we define a critical index z via the rescaling of time $t' = tb^{-z}$ where $b = 2$ is the space scale factor. From equation (2.7) linearised with respect to ω at the ferromagnetic fixed point we get $z = 2$. The significance of z is the same as in the continuum spin models (Hohenberg and Halperin 1977). The average magnetisation, $M(t) = \langle \Sigma_i q_i(t) \rangle / \Sigma_i 1$, is assumed to be time-dependent with characteristic time $\tau(T)$ such that $M(t, T) = \xi^{-\beta/\nu} m(t/\tau(T))$, where β and ν are the usual critical indices of the magnetisation and correlation length, respectively. Since in the present calculations the spins do not rescale, $M[\xi(l), t(l)] = M(\xi, t)$, and $\beta = 0$ (Nelson and Fisher 1975). However at the lth stage, $\xi(l) = b^{-l}\xi$, and $t(l) = tb^{-lz}$. We now choose l so that $(\xi(l))^{-1} = c$ ($c \ll 1$ so that the linear approximation of the recursion relation is valid). We obtain $M(\xi, t) = m(t/\xi^z)$ where $m(x) = M(c^{-1}, c^{-z}x)$. Thus $\tau \sim \xi^z$ and one identifies z as the dynamic exponent. We have not introduced z here in the framework of the time-dependent correlation function, as is usually done. This is due to the fact that we have discussed only the first moment of the master equation. A study of higher correlations leads, naturally, to the same z, since there is only one time scale as long as there are no more degrees of freedom except the order parameter, and we are asymptotically close to the critical point. The results we obtained are easily checked from the exact expressions (Glauber 1963):

$$M(t) = M(0)\exp[-(1-\gamma)t] \qquad \text{and} \qquad \langle M(0)M(t)\rangle \sim \exp[-(1-\gamma^2)|t|].$$

The asymptotic behaviour of $(1-\gamma^2)/4 \sim (1-\gamma)/2$ is ξ^{-2} which corresponds to $z = 2$.

The above value of z was obtained by the linearisation of equation (2.7). This step is not *a priori* correct and has to be justified. The dynamic process described by equation (2.2) is a Markoffian one (depending only on one time). Thus the static parameter space has to be enlarged by only one more parameter, the time scale, in order to describe the dynamics. After one iteration of the RG a term proportional to $(-\mathrm{i}\omega)^2$ is created, corresponding to a memory effect. Thus, in order to have a consistent RG we have to start with an initial parameter space which includes all the coefficients of the derivatives with respect to the time. We have to substitute, $\mathrm{d}/\mathrm{d}t$ in equation (2.7), $\mathrm{d}/\mathrm{d}t \to f(\mathrm{d}/\mathrm{d}t)$, such that $f(0) = 0$ and $\mathrm{d}f(x)/\mathrm{d}x|_0 = 1$.

The recursion relation (2.7) becomes an exact recursion relation for $f(-\mathrm{i}\omega)$

$$f'(-\mathrm{i}\omega) = f(-\mathrm{i}\omega)(2+f(-\mathrm{i}\omega))/(1-\tfrac{1}{2}\gamma^2). \tag{2.8}$$

The value of z will be determined by the demand that the bare time scale will be a marginal variable. Thus it is obtained by linearising equation (2.6), leading to the value $z=2$. It is easy to check that under this scaling the quadratic term has an exponent $-2z+2=-2$. Thus the quadratic term is irrelevant in the RG sense and does not change the previous results. These transient memory effects correspond to the short time scale found in the numerical calculations of Ma (1976a, b).

It is interesting to see what happens when the system is subject to an external magnetic field H. When $H\neq 0$, the Hamiltonian (2.1) includes a term $H\,\Sigma_i\,\sigma_i$ and the master equation becomes (Glauber 1963):

$$\mathrm{d}q_i(t)/\mathrm{d}t=-q_i(t)+B+\tfrac{1}{2}\gamma(q_{i-1}(t)+q_{i+1}(t))-\tfrac{1}{2}B\gamma(r_{i-1,i}(t)+r_{i,i+1}(t)) \tag{2.9}$$

where $r_{ij}=\Sigma_{\{\sigma\}}\,\sigma_i\sigma_jP(\{\sigma\},t)$ and $B=\tanh(H/k_\mathrm{B}T)$. The time behaviour of the r depends on higher order spin correlations, creating a hierarchy of equations. Glauber (1963) suggested that this set of equations be linearised with respect to $H/k_\mathrm{B}T$, assuming the deviations from equilibrium to be proportional to H. Under this approximation, we can substitute $r_{i-1,i}(t)$ by its equilibrium value $Q=r_{i-1,i}(\infty)=r_{i,i+1}(\infty)$. Equation (2.9) becomes

$$\mathrm{d}q_i/\mathrm{d}t=-q_i+\tfrac{1}{2}\gamma(q_{i-1}+q_{i+1})+b \qquad \text{where } b=B(1-\gamma Q). \tag{2.10}$$

The decimation of this model shows that equations (2.6)–(2.7) are still the recursions for γ and the time scale. This result is consistent with the exact static recursions (Nelson and Fisher 1975) which do not include terms linear in H. It also means that z has the same value as the $H=0$ case. The recursion relation for the constant term is

$$b'=2b(1+\gamma)/(2-\gamma^2). \tag{2.11}$$

Equation (2.11) comes from two recursion relations; one is for $1-\gamma Q$ and the other for B. These can be obtained either from the known value of $Q=[1-(1-\gamma^2)^{1/2}]/\gamma$ or by direct decimation of the equation of motion of the r. The result is:

$$(1-\gamma Q)'=2(1-\gamma Q)/(2-\gamma^2). \tag{2.12}$$

By substituting into equation (2.11) we obtain

$$B'=B(1+\gamma)$$

which agrees with the static recursion relation linearised with respect to B (Nelson and Fisher 1975).

3. The block transformation

The decimation procedure, which was described in the previous section has the disadvantage of assuming that $\eta=2-d$. This assumption which is justified in $d=1$, becomes worse when d increases hence preventing one from reaching the right fixed point (Wegner 1976). This difficulty does not arise in the block transformation which describes a set of spins in a block by new spin variable, and introduces a rescaling factor (Niemeijer and van Leeuwen 1976, Kadanoff *et al* 1976). The reason for preferring a certain form of $\mathscr{S}\{\mu,\sigma\}$ (1.1) is discussed in detail by Nelson and Fisher (1975), and will not be repeated here. In particular, the transformation,

$$\mathscr{S}\{\mu,\sigma\}=\prod_i\mathscr{S}_i\{\mu,\sigma\}=\prod_i\exp[\mu_i(p_1\sigma_{2i}+p_2\sigma_{2i+1})]/2\cosh(p_2\sigma_{2i}+p_2\sigma_{2i+1}) \tag{3.1}$$

associates a new spin variable, μ_i, with a block of two adjacent spins σ_{2i} and σ_{2i+1}. In order that the transformed Hamiltonian remains in the original parameter space of nearest neighbour coupling, the p_i are not independent and restricted by (Nelson and Fisher 1975):

$$\exp(2J/k_B T) = \cosh(p_1 + p_2)/\cosh(p_1 - p_2). \tag{3.2}$$

Quantities which depend on μ_i can be related to those of σ_i by:

$$\sum_{\mu_i=\pm 1} \mu_i \mathscr{S}_i(\mu_i; \sigma_{2i}, \sigma_{2i+1}) = \tanh(p_1\sigma_{2i} + p_2\sigma_{2i+1}) = A_1\sigma_{2i} + A_2\sigma_{2i+1} \tag{3.3}$$

where $A_{\frac{1}{2}} = \frac{1}{2}[\tanh(p_1+p_2) \pm \tanh(p_1 - p_2)]$. Relation (3.3) has to be understood as subject to further tracing over $\{\sigma\}$. Using (3.3) and (2.4) we can derive the equation of motion of the spin block $\langle\mu_i\rangle$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\mu_i\rangle = -\langle\mu_i\rangle + \tfrac{1}{2}\gamma(A_1\langle\sigma_{2i-1} + \sigma_{2i+1}\rangle + A_2\langle\sigma_{2i} + \sigma_{2(i+1)}\rangle. \tag{3.4}$$

In order to eliminate the $\{\sigma\}$ from the right-hand side of (3.4) in favour of $\{\langle\mu_i\rangle\}$, we define a complimentary spin-block variable ζ_i, to span all degrees of freedom of the system before the renormalisation. Choosing

$$\begin{aligned}\mathscr{S}'\{\zeta, \sigma\} &= \prod_i \mathscr{S}'_i(\zeta_i; \sigma_{2i-1}, \sigma_{2i}) \\ &= \prod_i \exp[\zeta_i(p_2\sigma_{2i-1} + p_2\sigma_{2i})]/2\cosh(p_i\sigma_{2i-1} + p_2\sigma_{2i}),\end{aligned} \tag{3.5}$$

equation (3.4) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\mu_i\rangle = -\langle\mu_i\rangle + \tfrac{1}{2}\gamma(\langle\zeta_i\rangle + \langle\zeta_{i+1}\rangle). \tag{3.6}$$

A similar equation of motion is obeyed by the $\{\langle\zeta_i\rangle\}$. Having now two equivalent sublattices, the rest of the calculation is similar to the decimation which was described in § 2.

The Fourier components of the block spins are related by

$$\langle\zeta_i\rangle = \tfrac{1}{2}\gamma(\langle\mu_{i-1}\rangle + \langle\mu_i\rangle)/(1 - \mathrm{i}\omega). \tag{3.7}$$

Substituting back into (3.6) we obtain the recursion relations (2.6)–(2.7). As in the static case (Nelson and Fisher 1975) the fixed point of the RG transformation does not determine p_1 and p_2 separately, but only constrains them by equation (3.2). Thus we have a continuum set of RG transformations which have, of course, the same dynamics.

We shall conclude the paper with a remark concerning the conventional theory (Van Hove 1954) which predicts $z_{\text{conv}} = 2 - \eta = 1$. The $z = 2$, we found here, can be obtained of course from the exact solution (Glauber 1963), but we think that the large disagreement between z and z_{conv} worth mentioning.

Acknowledgments

I wish to thank J M Kosterlitz and D J Thouless for illuminating discussions.

References

Ferrell R A, Monyhard N, Schmidt H, Schwabl F and Szepfalusy P 1968 *Ann. Phys., NY* **47** 565
Glauber R J 1963 *J. Math. Phys.* **4** 294
Halperin B I and Hohenberg P C 1969 *Phys. Rev.* **177** 952
Halperin B I, Hohenberg P C and Ma S 1972 *Phys. Rev. Lett.* **29** 1548
—— 1974 *Phys. Rev.* B **10** 139
Hohenberg P C and Halperin B I 1977 *Rev. Mod. Phys.* **49** 435
Kadanoff L P, Houghton A and Yalabic M C 1976 *J. Statist. Phys.* **14** 171
Ma S 1976a *Modern Theory of Critical Phenomena* (New York: Benjamin)
—— 1976b *Phys. Rev. Lett.* **37** 461
Myerson R J 1976 *Phys. Rev.* B **14** 4136
Nelson D R and Fisher M E 1975 *Ann. Phys., NY* **91** 226
Niemeijer T and Van Leeuwen J M J 1976 *Phase Transitions and Critical Phenomena* vol. 6, eds C Domb and M S Green (New York: Academic) p 425
Van Hove L 1954 *Phys. Rev.* **93** 1374
Wegner F 1976 *Phase Transitions and Critical Phenomena* vol. 6, eds C Domb and M S Green (New York: Academic) p 29
Wilson K G 1971 *Phys. Rev.* B **4** 3174, 3184
Wilson K G and Fisher M E 1972 *Phys. Rev. Lett.* **28** 240
Wilson K G and Kogut J 1974 *Phys. Rep.* **12** 75